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B70 09076

SUBJECT: Potential Expansion for a Non-Homogeneous Oblate Spheroid  
Case 310

DATE: September 29, 1970

FROM: S. L. Levie, Jr.

ABSTRACT

The coefficients are computed for the spherical harmonic expansion of the potential of an oblate spheroid whose density is an arbitrary function of equatorial radius. The density is assumed constant on oblate spheroidal surfaces whose eccentricities are the same as that of the given spheroid. The solution begins with the known potential expansion of a homogeneous oblate spheroid and develops the expansion valid for an arbitrary number of oblate spheroidal layers of different densities surrounding an oblate spheroidal core. By a limiting process, this expansion is extended to the result just stated. A convergence criterion is obtained, the result's significance is explained, and an extension to the case of a mascon is made.

(NASA-CR-114206) POTENTIAL EXPANSION FOR A  
NON-HOMOGENECUS OBLATE SPHEROID, CASE 310  
(Bellcomm, Inc.) 16 p

N79-73189

Unclas

00/64 12808

FF No. 602(C)	(PAGES)	(CODE)
	CR-114206	24
	(NASA CR OR TMX OR AD NUMBER)	(CATEGORY)



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MEMORANDUM FOR FILE

INTRODUCTION AND SUMMARY

In potential theory it is sometimes of interest to determine a mass distribution whose gravitational potential in some sense [1] provides a good representation of a given potential function. A useful mass distribution for many purposes is an oblate spheroid of equatorial radius  $a$  and eccentricity  $e$  whose density  $\rho$  varies with equatorial radius and is constant on surfaces of eccentricity  $e$ . With  $a$ ,  $e$ , and  $\rho$  suitably chosen, such a distribution should give a very good representation of the earth's potential and a fairly good representation of the moon's potential. Furthermore, such a distribution can provide a convenient description of mascons on either planet.

If the non-homogeneous oblate spheroidal mass distribution is to have any utility, an expression is required for its external potential in its symmetry frame.\* Such an expression is provided in this paper. Starting from the known expansion in spherical harmonics of the potential of a homogeneous oblate spheroid [2], a formula is obtained for the coefficients in the potential expansion for a body consisting of homogeneous oblate spheroidal shells of different densities surrounding a homogeneous oblate spheroidal core. The arrangement is such that all the oblate spheroidal surfaces involved have the same eccentricity  $e$ , the same equatorial plane, and a common symmetry axis. By itself, such a layered spheroid is useful in modeling planetary potentials. By a limiting process, this expansion is converted to a form valid for an oblate spheroid of eccentricity  $e$  whose density is a continuous function of equatorial radius and is constant on oblate spheroidal surfaces of eccentricity  $e$ . It is shown that all the potential expansions presented converge in the region  $r > ae$ , where  $a$  is the equatorial radius of any of the figures. However, these series represent a potential only at points external to the body.

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\*In the symmetry frame, the  $x$ - and  $y$ - axes lie in the equatorial plane and the  $z$ -axis coincides with the symmetry axis of the spheroid.

Formulas are provided for the masses of the bodies discussed. However, since all the potential expansions are normalized with respect to mass, a mass assignment may be made in the potentials without the necessity of determining the implied scaling factor in the density function.

An interesting mathematical observation is made connecting the expansion coefficients derived for the continuous-density, oblate spheroidal figure with the general formula available from potential theory. It is seen that the limiting procedure just described is equivalent to unlocking a difficult double integral.

Because the shapes and mass distributions of the figures under consideration are plausible and mathematically simple candidates for mascons (or mass concentrations) which exist on the earth and moon, a formula is given which expresses the potentials of these figures when displaced from the origin. The displacement is allowed to be arbitrary, but the figure's symmetry axis is assumed to coincide with the radius vector to the figure's center. In a geographic coordinate system, for example, this formula may be used with one of the expansions just discussed to obtain, by superposition, the total potential of an oblate spheroidal earth with an oblate spheroidal mascon inside.

The development, which is presented in a logical sequence of simple steps, begins in the next section.

#### HOMOGENEOUS OBLATE SPHEROID

MacMillan [2] presents the potential function valid at all points outside a homogeneous oblate spheroid and then writes the expansion of this function in spherical harmonics. The coordinate system in which the potential is written has its x-y plane in the equatorial plane of the spheroid and its z-axis coincident with the spheroid's symmetry axis. This expansion, in a form suitable for geodesy and selenodsy, is

$$U_{\text{Spheroid}}(r, \theta, \phi) = \frac{GM}{r} \sum_{k=0}^{\infty} D_k \left( \frac{a}{r} \right)^k P_k(\cos \theta) \quad (1)$$

where

$$D_k = \begin{cases} 3(-)^{k/2} e^k / ((k+1)(k+3)) & (k = \text{even}) \\ 0 & (k = \text{odd}). \end{cases} \quad (2)$$

In these equations  $a$  and  $e$  are the equatorial radius and eccentricity of the oblate spheroid, and  $(r, \theta, \phi)$  are the spherical polar coordinates of the field point at which the potential is computed. Notice that because of the symmetry of the spheroid the longitude  $\phi$  does not appear in the potential.  $P_k$  denotes Legendre's polynomial,  $G$  the universal gravitational constant, and  $M$  the spheroid's mass, which is related to the density  $\rho$  by

$$M_{\text{Spheroid}} = \frac{4}{3} \pi \rho \sqrt{1-e^2} a^3. \quad (3)$$

The potential coefficients (expansion coefficients)  $D_k$  depend only on the eccentricity of the oblate spheroid, if it is agreed to put the factor  $a^k$  in the potential formula (1). Although this separation is arbitrary, it is common practice to write the potential in terms of the factor  $(a/r)^k$ . With one exception, all the potentials written in this paper have the form (1), and their expansion coefficients are proportional to (2). The exception is the mascon's potential, which is not referred to a frame of symmetry.

Although the function represented by (1) is valid only for points outside a homogeneous oblate spheroid, (1) itself is convergent for points satisfying  $r > ae$ , some of which lie inside the spheroid. Thus one must be careful not to apply (1) to points inside the spheroid nor to points outside the spheroid for which  $r < ae$ . This is to say that (1) converges to the potential of a homogeneous oblate spheroid at all points outside the spheroid for which  $r > ae$ .

HOMOGENEOUS OBLATE SPHEROIDAL SHELL

The potential of a homogeneous oblate spheroidal shell is the key to the development in this paper. The figure under consideration is illustrated in Figure 1. The shell is a homogeneous mass distribution of density  $\rho_1$  bounded on the outside by an oblate spheroidal surface of equatorial radius  $a_1$  and eccentricity  $e$  and bounded on the inside by an oblate spheroidal surface of equatorial radius  $a_2$  and the same eccentricity  $e$ . The surface and the coordinate system possess the symmetries illustrated.

For the purposes of potential theory, this shell may be imagined to be a superposition of two homogeneous oblate spheroids, the larger of positive mass and the smaller of negative mass, and both of the same density. Thus the mass of the shell illustrated may be found from

$$\begin{aligned}
 M_{\text{Shell}} &= M_{a_1} + M_{a_2} \\
 &= \frac{4}{3} \pi \rho_1 \sqrt{1-e^2} a_1^3 \\
 &\quad - \frac{4}{3} \pi \rho_1 \sqrt{1-e^2} a_2^3 \\
 &= \frac{4}{3} \pi \rho_1 \sqrt{1-e^2} (a_1^3 - a_2^3)
 \end{aligned} \tag{4}$$

where (3) has been used. Likewise, the potential of the shell may be found by summing

$$U_{a_1} = \frac{GM_{a_1}}{r} \sum_{k=0}^{\infty} D_k \left( \frac{a_1}{r} \right)^k P_k(\cos \theta) \tag{5}$$

and

$$U_{a_2} = \frac{GM_{a_2}}{r} \sum_{k=0}^{\infty} D_k \left( \frac{a_2}{a_1} \right)^k \left( \frac{a_1}{r} \right)^k P_k(\cos \theta), \tag{6}$$

where (1) has been used.

Remembering that  $M_{a_2}$  is negative, the resulting potential may be shown to be

$$U_{\text{Shell}}(r, \theta, \phi) = \frac{GM_{\text{Shell}}}{r} \sum_{k=0}^{\infty} C_k^{\text{Shell}} \left(\frac{a_1}{r}\right)^k P_k(\cos \theta), \quad (7)$$

where

$$C_k^{\text{Shell}} = \frac{D_k}{a_1^k} \frac{a_1^{k+3} - a_2^{k+3}}{a_1^3 - a_2^3}. \quad (8)$$

In the limit of a very thin shell, for which  $a_2 = a_1 - t$ , where  $t$  is very small, then

$$M_{\text{Shell}} \rightarrow 4\pi\rho_1 a_1^2 t \sqrt{1 - e^2} \quad (9)$$

and

$$C_k^{\text{Shell}} \rightarrow \left(1 + \frac{k}{3}\right) D_k. \quad (10)$$

It may be observed that the coefficients for the thin shell are larger by a factor  $(1 + k/3)$  than those of a homogeneous oblate spheroid. If the two figures have the same size and mass, which requires a much greater density in the shell, then the shell's potential is correspondingly larger everywhere, since the mass is all at the surface.

#### SPHEROID BOUNDED BY SPHEROIDAL SHELL

Suppose that in the previous section the cavity within the shell is filled with a homogeneous distribution of mass at density  $\rho_2$ . The body obtained is illustrated in Figure 2. As before, all surfaces involved have the same eccentricity  $e$ .

The potential of this composite body may be obtained by superposing the contributions of the shell and the internal spheroid. Referring to formulas (7) and (1), this means adding the two potentials

$$U_{\text{Shell}} = \frac{GM_{\text{Shell}}}{r} \sum_{k=0}^{\infty} C_k^{\text{Shell}} \left( \frac{a_1}{r} \right)^k P_k(\cos \theta)$$

and

$$U_{\text{Spheroid}} = \frac{GM_{\text{Spheroid}}}{r} \sum_{k=0}^{\infty} D_k \left( \frac{a_2}{a_1} \right)^k \left( \frac{a_1}{r} \right)^k P_k(\cos \theta).$$

The second formula has been arranged to display the equatorial radius of the outermost surface. Performing the addition and utilizing (3), (4) and (8) gives the result

$$U^{(2)}(r, \theta, \phi) = \frac{GM^{(2)}}{r} \sum_{k=0}^{\infty} C_k^{(2)} \left( \frac{a_1}{r} \right)^k P_k(\cos \theta) \quad (11)$$

where

$$C_k^{(2)} = \frac{D_k}{a_1^k} \frac{\rho_1 (a_1^{k+3} - a_2^{k+3}) + \rho_2 a_2^{k+3}}{\rho_1 (a_1^3 - a_2^3) + \rho_2 a_2^3} . \quad (12)$$

The superscripts in these equations are intended to indicate that a nesting of two different homogeneous bodies is involved. The total mass of the body is

$$M^{(2)} = \frac{4}{3}\pi \sqrt{1 - e^2} \left[ \rho_1 (a_1^3 - a_2^3) + \rho_2 a_2^3 \right] . \quad (13)$$

Notice that these formulas reduce to those just presented for the case of a hollow homogeneous shell when  $\rho_2 = 0$ , and that they reduce to the homogeneous spheroid formulas when  $\rho_2 = \rho_1$ .

#### SPHEROID BOUNDED BY (n-1) SPHEROIDAL SHELLS

The generalization of the formulas of the last section to the case of (n-1) homogeneous oblate spheroidal shells surrounding a homogeneous oblate spheroid is straightforward. There are n bodies, n densities, and n surfaces involved in this

composite figure. The surfaces are all oblate spheroids of the same eccentricity  $e$ , and they share a common equatorial plane and a common symmetry axis. Working from the outside in, the bodies are serially assigned integer names, beginning with "1" and ending with "n". Thus the outer shell has density  $\rho_1$  and the inner spheroid has density  $\rho_n$ . The surfaces are numbered likewise, so the outer surface has equatorial radius  $a_1$  and the inner surface  $a_n$ . By construction we have

$$a_i > a_{i+1}$$

for all  $i$  in the interval  $(1, n-1)$ .

With these conventions in hand, the generalization of the pervious formulas is

$$U^{(n)}(r, \theta, \phi) = \frac{GM^{(n)}}{r} \sum_{k=0}^{\infty} C_k^{(n)} \left( \frac{a_1}{r} \right)^k P_k(\cos \theta) \quad (14)$$

where

$$C_k^{(n)} = \frac{D_k}{a_1^k} \frac{\rho_n a_n^{k+3} + \sum_{i=1}^{n-1} \rho_i (a_i^{k+3} - a_{i+1}^{k+3})}{\rho_n a_n^3 + \sum_{i=1}^{n-1} \rho_i (a_i^3 - a_{i+1}^3)} \quad (15)$$

The mass of the body is

$$M^{(n)} = \frac{4}{3}\pi \sqrt{1 - e^2} \left[ \rho_n a_n^3 + \sum_{i=1}^{n-1} \rho_i (a_i^3 - a_{i+1}^3) \right] \quad (16)$$

#### SPHEROID WITH DENSITY AS CONTINUOUS FUNCTION OF EQUATORIAL RADIUS

In this section the spherical harmonic expansion of the potential of an oblate spheroid of continuously varying density is obtained. The method is to take the limit of the previous formulas as  $n$  tends to infinity and as the equatorial radius of the inner spheroid tends to zero. By virtue of the previous construction, the series obtained will be valid for a density function which varies with equatorial radius within the oblate spheroid and is constant on surfaces whose eccentricities are the same as that of the surface of the given spheroid.



Let the given oblate spheroid have equatorial radius  $a$ , so that  $a_1 = a$ , and let its surface have eccentricity  $e$ . Setting  $a_n = 0$ , let  $n \rightarrow \infty$  in (14), (15) and (16). Dropping superscripts, the result is

$$U(r, \theta, \phi) = \frac{GM}{r} \sum_{k=0}^{\infty} C_k \left( \frac{a}{r} \right)^k P_k(\cos \theta) \quad (17)$$

where

$$C_k = \frac{D_k}{a^k} \frac{\int_0^a \rho(x) d(x^{k+3})}{\int_0^a \rho(x) d(x^3)}, \quad (18)$$

in which the integration variable  $x$  is the equatorial radius within the oblate spheroid. The spheroid's total mass is

$$M = \frac{4}{3}\pi \sqrt{1 - e^2} \int_0^a \rho(x) d(x^3). \quad (19)$$

Equation (18) may be checked by setting  $\rho = \text{constant}$ , simulating a homogeneous oblate spheroid. It may be verified that  $C_k = D_k$  in this case, as expected.

A very important point may be observed by comparing (18) and (19). It is that the expansion coefficients  $C_k$  are automatically normalized with respect to mass. The significance of this is that when writing the potential (17) a number may be inserted for  $M$  without the need for scaling the density function to make (19) yield that number. In effect, (18) accounts for the scaling automatically. These remarks apply to all the potentials and expansion coefficients written in this paper.

#### CONVERGENCE

None of the infinite series presented thus far is usable if the convergence of the series is not known. The question of convergence of these series is simple to answer,

provided it is realized that the potential of each configuration considered is the sum of potentials of homogeneous oblate spheroids (approximately half of which have negative mass). Hence, for any of the given series to converge, it is only necessary that each of the constituent series converges. It follows from remarks in the section on the homogeneous oblate spheroid that the condition  $r > ae$  is sufficient for the convergence of each constituent series, and therefore for the given series as well.

A second important question is that of representation: under what conditions does one of the convergent series represent the actual potential of a body. Referring again to the section about the homogeneous oblate spheroid and employing an argument similar to the one just used, it is concluded that when a series converges ( $r > ae$ ) it represents the potential only at points lying outside the body.

#### SIGNIFICANCE OF THE DEVELOPMENT

It can be shown [3] that the general spherical harmonic expansion of the potential of an axially symmetric body is

$$U(r, \theta, \phi) = \frac{GM}{r} \sum_{k=0}^{\infty} C_k \left(\frac{a}{r}\right)^k P_k(\cos \theta) \quad (20)$$

where

$$C_k = \frac{2\pi}{a^k M} \iint_{\text{body}} d\xi \, d(\cos \alpha) \, \rho(\xi, \alpha) \, \xi^{k+2} P_k(\cos \alpha). \quad (21)$$

In this formula, which is written in a frame whose z-axis coincides with the body's symmetry axis,  $(r, \theta, \phi)$  are the spherical polar coordinates of the field point,  $(\xi, \alpha)$  are the radius to and colatitude of a symmetric differential ring of mass,  $\rho(\xi, \alpha)$  is the density function,  $a$  is a size parameter to be specified, and  $M$  is the total mass. Figure 3 illustrates the geometry for the case of an oblate spheroid of eccentricity  $e$ , with the further specialization that the x-y plane coincides with the equatorial plane. The length parameter  $a$  is now taken as the equatorial radius. Suppose it is desired to specify that the density

$\rho(\xi, \alpha)$  be constant on oblate spheroidal shells of eccentricity  $e$  and depend on equatorial radius only. Formula (21) is the general rule for computing the potential coefficients for this problem, yet it is not obvious how to perform the required computation.

The significance of the development just completed is that it sidesteps this problem and arrives at the coefficients by a different route. Equating (18) and (21), and using (19), the difficult double integral in (21) can be specialized to

$$\begin{aligned} \iint_{\text{body}} d\xi \, d(\cos\alpha) \rho(\xi, \alpha) \xi^{k+2} P_k(\cos\alpha) \\ = 2 \left(1 + \frac{k}{3}\right) D_k \sqrt{1 - e^2} \int_0^a x^2 \rho(x) dx, \end{aligned} \quad (22)$$

in which  $\rho(\xi, \alpha)$  and  $\rho(x)$  represent the same density function. It is interesting to observe in (22) that the coefficients for a thin shell,  $(1 + k/3)D_k$ , show up multiplied by a factor which accounts for the density variation from shell to shell.

#### NON-HOMOGENEOUS OBLATE SPHEROIDAL MASCON

The oblateness of the mass distributions that have been discussed suggests a plausible shape for a mascon envisioned as lying inside a planet with the mascon's symmetry axis aligned with the planetographic vector  $\bar{R}$  locating its center. This configuration is shown in Figure 4, in which  $(R, \theta_M, \phi_M)$  are the spherical polar coordinates of  $\bar{R}$ . The spherical harmonic expansion of the potential of an oblate spheroid with this location and orientation is derived in [4]. It is

$$U(r, \theta, \phi) = \frac{GM}{r} \sum_{k=0}^{\infty} \sum_{m=0}^k \left(\frac{a}{r}\right)^k P_k^m(\cos\theta) [C_{km} \cos m\phi + S_{km} \sin m\phi] \quad (23)$$

where

$$\begin{pmatrix} C_{km} \\ S_{km} \end{pmatrix} = \sum_{\ell=0}^k C_{\ell} (2-\delta_{m0}) \frac{(k-m)!}{(k+m)!} \frac{k!}{\ell!(k-\ell)!} \cdot \left(\frac{R}{a}\right)^{k-\ell} P_k^m(\cos\theta_M) \begin{pmatrix} \cos m\phi_M \\ \sin m\phi_M \end{pmatrix}, \quad (24)$$

in which  $a$  is the mascon's equatorial radius,  $M$  its mass,  $(r, \theta, \phi)$  the field point in planetographic coordinates, and  $P_k^m(\cos\theta)$  the unnormalized associated Legendre polynomial as defined in [3], [4], [5], or [6]. The coefficients  $C_{\ell}$  depend on the mascon's equatorial radius, eccentricity, and density function. If the mascon's density is assumed to be any of the layered functions discussed herein, the corresponding expansion coefficients may be used for  $C_{\ell}$ . (Otherwise, the general formula (21) must be used for  $C_{\ell}$ .) For these densities, the series (23) converges to the potential function of the mascon at all points which are outside the mascon and outside a sphere of radius  $a$  centered at the mascon's center. In addition, the convergence of (23) dictates the general requirement  $r > a$ , which is mentioned in [4].

Using superposition, the mascon potential (23) may be added to a function (one of those just presented, for example) representing the gross potential of a planet to obtain the potential of a "lumpy" planet. This may be repeated until a physically plausible mass distribution has been obtained for the planet. By comparison with the known potential, the resulting potential expansion may be used to determine the validity of the assumed mass distribution or to adjust various assumed parameters appearing in it. If it is desired to adjust parameters, the various masses should be taken as a parameter set first, since the masses appear linearly in the composite potential.

*Sterling Levie Jr.*  
S. L. Levie, Jr.

2014-SLL-bsb

Attachments

# BELLCOMM, INC.

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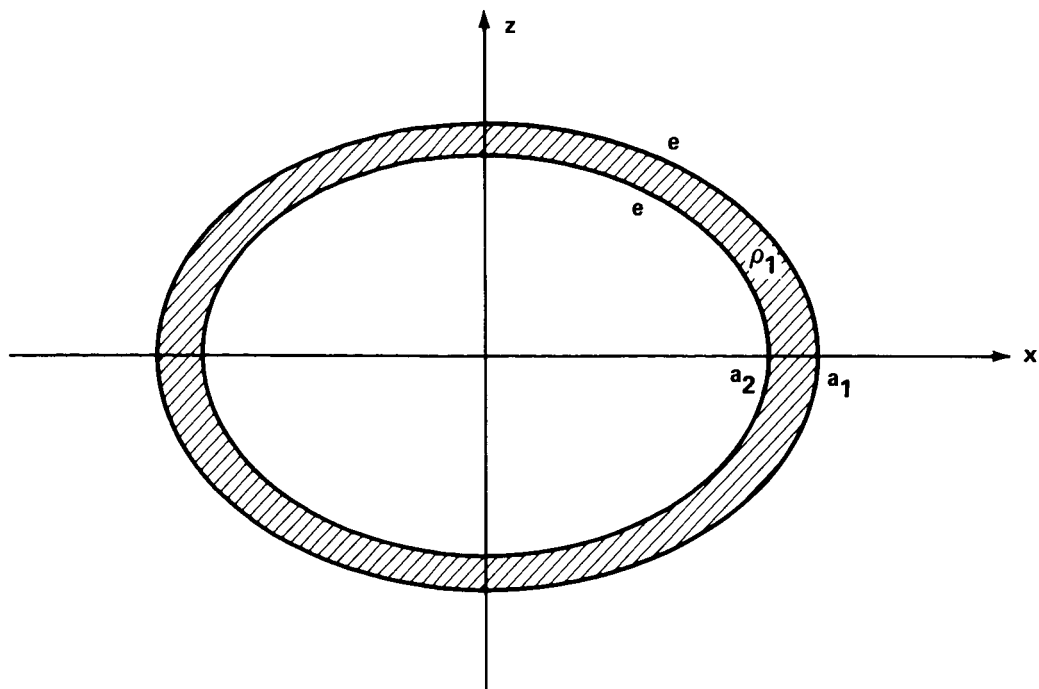


FIGURE 1 - HOMOGENEOUS OBLATE SPHEROIDAL SHELL.

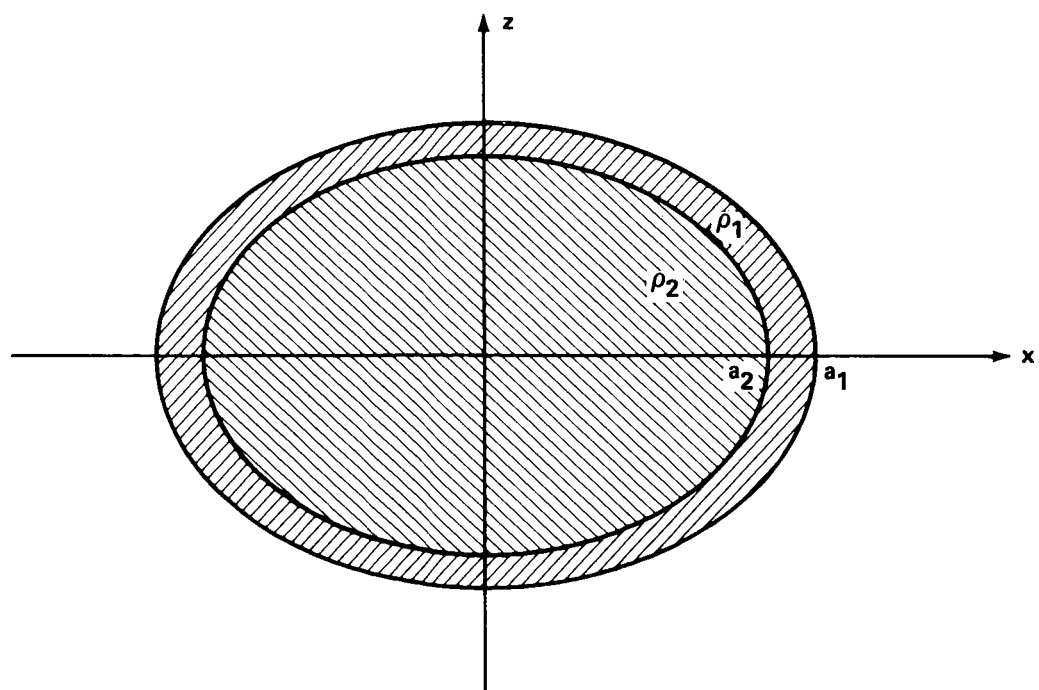
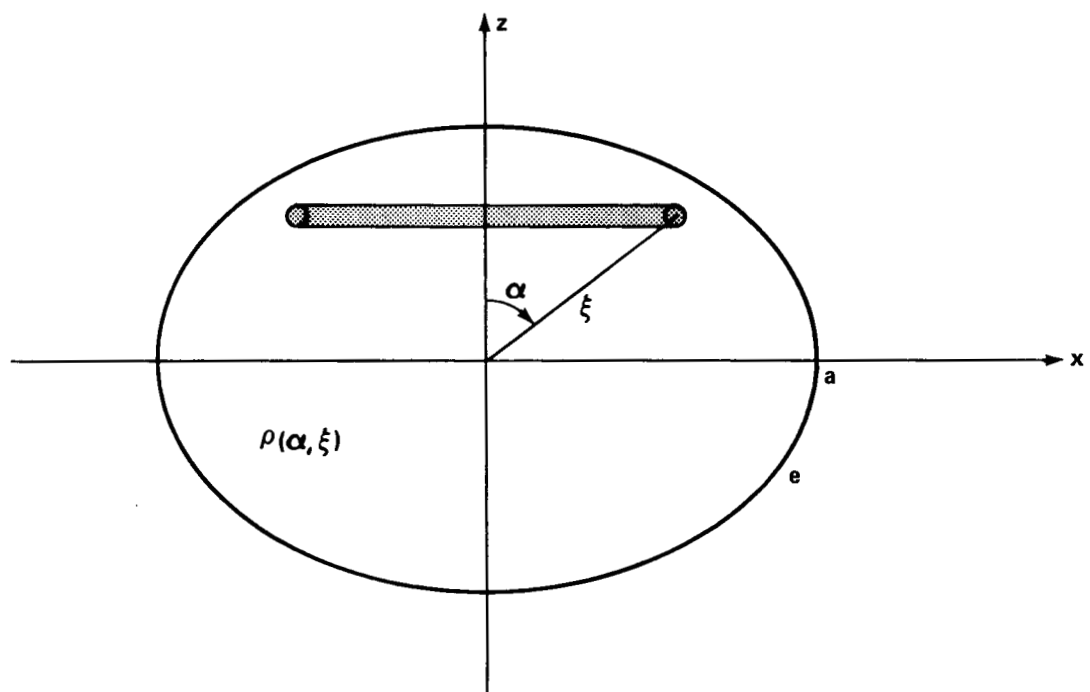


FIGURE 2 - HOMOGENEOUS OBLATE SPHEROID PLUS HOMOGENEOUS OBLATE SPHEROIDAL SHELL.



**FIGURE 3 - NON-HOMOGENEOUS OBLATE SPHEROID SHOWING SYMMETRIC DIFFERENTIAL RING.**

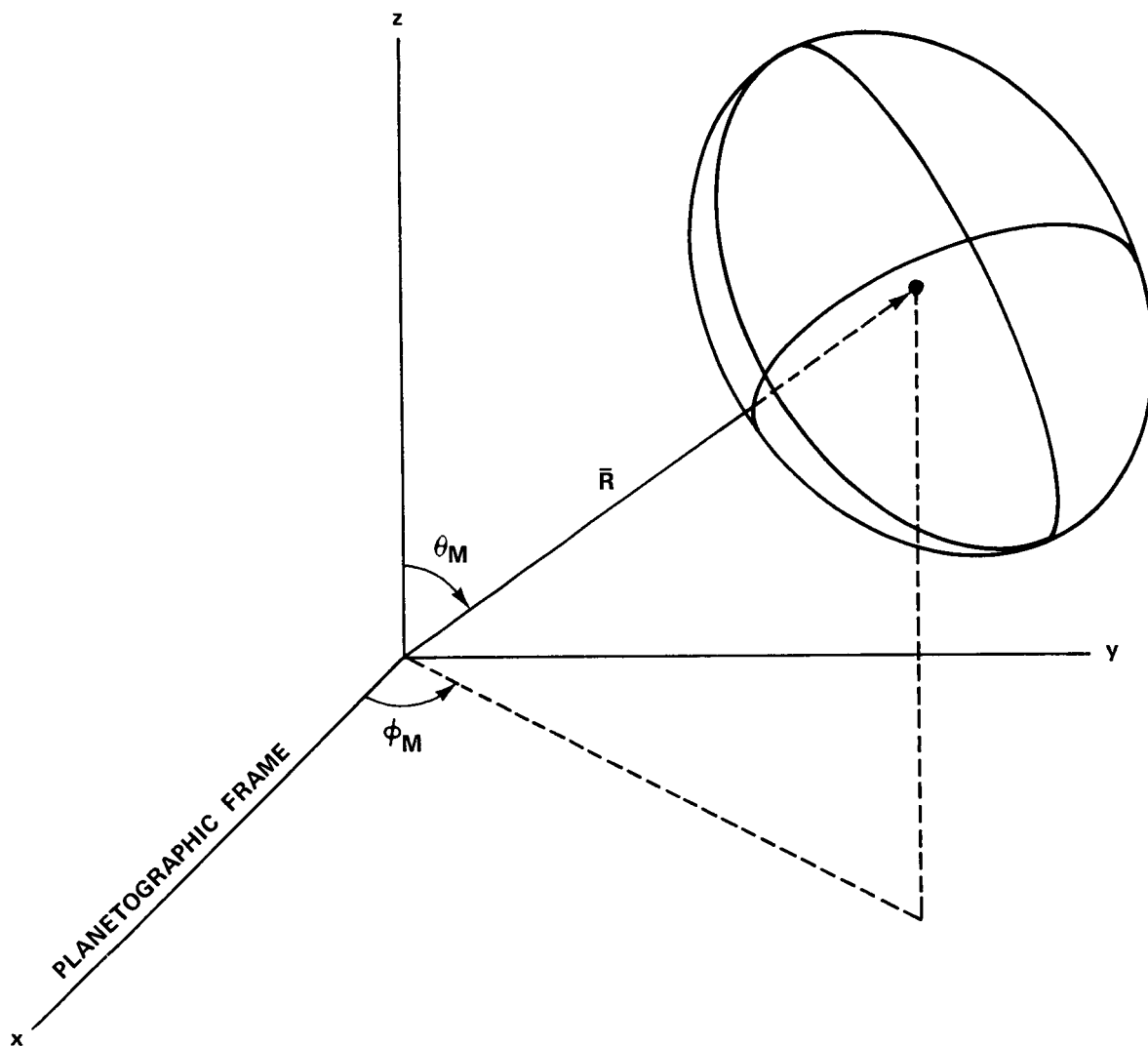


FIGURE 4 — NON-HOMOGENEOUS OBLATE SPHEROIDAL MASCON AT  $(R, \theta_M, \phi_M)$  IN PLANETOGRAPHIC FRAME. NOTE MASCON'S AXIAL SYMMETRY WITH RESPECT TO  $\bar{R}$ .